PAPER CODE
PHYS393

THE UNIVERSITY
of LIVERPOOL

JANUARY 2007 EXAMINATIONS

Bachelor of Science: Year 3
Master of Physics: Year 3
Master of Physics: Year 4

STATISTICAL AND LOW TEMPERATURE PHYSICS

TIME ALLOWED: THREE HOURS

INSTRUCTIONS TO CANDIDATES

Answer **all** questions.

Question 1 carries 50% of the total marks.

Question 2 and 3 each carry 25% of the total marks.

The marks allotted to each part of a question are indicated in square brackets.

In the event of a student answering both parts of an either/or question and not clearly crossing out one answer, only the answer to part (a) of the question will be marked.

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1.

(a) A set of 5 distinguishable particles can occupy energy states 0, ε, 2ε, 3ε, 4ε - the total energy of the set is 4ε.

(i) Write out the five possible distributions of the particles in the energy states **[2]**

(ii) Enumerate the number of microstates for each distribution **[2]**

(iii) Evaluate the mean population for each energy state. **[2]**

iv) If , instead of being distinguishable, the particles had been indistinguishable bosons, evaluate the mean population of each energy state. **[2]**

(b) In a solid state system the atoms can exist in energy states 0, ε, or 2ε where $\varepsilon = 1.38 \times 10^{-21}$ J.

Evaluate the temperature T in the different cases:

(i) The population of the state of energy ε is 0.1 times that of energy state 0. **[2]**

(ii) The population of the state of energy 2ε is 10^{-6} times that of the state of energy 0. **[2]**

(iii) The population of the state of energy 2ε is 0.99 that of the state of energy ε. **[2]**

(c) N atoms bound in a solid system at temperature T can exist in levels of energy 0 and ε. The level of energy 0 is a single state while the level of energy ε contains two states (has a degeneracy of 2).

(i) Write an expression for the Partition function Z of such an atom. **[2]**

(ii) Using the bridge relation

$$U = NkT^2 \frac{\partial(\ln Z)}{\partial T}$$

or otherwise, show that the internal energy U can be written as

$$U = \frac{2N\varepsilon.\exp(-\varepsilon/kT)}{[1 + 2\exp(-\varepsilon/kT]}$$ [2]

where k is the Boltzmann constant.

(iii) Derive the limits of U as $T \to 0$ and as $T \to \infty$. [2]

(iv) Sketch the variation of U with T. [2]

(v) Without further differentiation sketch the graph of heat capacity C_V versus T. [2]

(d) Describe what is meant by the Fermi energy of a system of electrons. [3]

In a system of N conduction electrons contained in a volume V, the density of states g(k) in terms of the wavevector k is given by

$$g(k)dk = 2. \frac{V.4\pi k^2 dk}{(2\pi)^3}$$

(i) Explain the origin of the 2 in the top line of the above expression. [1]

(ii) Show that at temperature T = 0K the k value at the Fermi surface, k_F, is given by

$$k_F = (3\pi^2 N/V)^{1/3}$$ [3]

(iii) Write an expression relating k_F to the Fermi energy ε_F [1]

(iv) Assuming that conduction electrons in metallic sodium (Na) can be represented by the theory above and that each Na atom releases one electron to the conduction band, evaluate the Fermi energy of sodium. [2]

Relative atomic mass of sodium = 23.0

Density of sodium = 971 kgm^{-3}.

(e) (i) Explain what is meant by Bose condensation. [3]

(ii) A system of N atoms of liquid He^4 contained in a volume V has a Bose condensation temperature T_B given by

$$T_B = \left(\frac{h^2}{2\pi mk}\right)\cdot\left(\frac{N}{2.612V}\right)$$

where h is the Planck constant, k is the Boltzmann constant and m the mass of an atom.

Evaluate T_B for liquid He^4 which has a molar volume of 27×10^{-6} m^3. [2]

(iii) Sketch a pressure versus temperature phase diagram for He^4. [2]

(iv) Identify on the above phase diagram the feature that can be associated with a Bose condensation. [2]

(f) State the nature of the current carriers in

(i) a superconducting material [1]

(ii) a normal conductor. [1]

Sketch a graph of resistivity ρ versus temperature T over the range 0K → 300K for

(iii) a good metallic conductor (Cu) [1]

(iv) a metallic superconductor (Nb) [1]

(v) a high temperature superconductor ($YBa_2Cu_3O_{7-x}$) [1]

(vi) On each graph indicate the nature of the current carriers [2]

2. Answer **either** 2(a) **or** 2(b)

(a) A system of a classical gas consists of N monatomic particles in a volume V. The distribution of states g(k) in terms of the wavevector k is written as

$$g(k)dk = \frac{V.4\pi k^2 dk}{(2\pi)^3}$$

(i) Write expressions relating particle speed v to the wavevector k, relating the speed v to particle energy ε and relating k to ε. **[3]**

(ii) Show that the distribution of states in terms of particle speed g(v) can be written as

$$g(v)\,dv = 4\pi V(m/h)^3 v^2 dv$$ **[3]**

where m is the mass of the particles and h is Planck's constant.

The probability f(ε) that a state of energy ε will be populated is given by

$$f(\varepsilon) = \frac{N}{Z}.\exp(-\varepsilon/k_B T)$$

where the Partition function Z is defined as

$$Z = \int_0^\infty g(k)\exp(-\varepsilon/k_B T)dk$$

and k_B is the Boltzmann constant.

(iii) Show that the Partition function Z can be expressed as

$$Z = V.(\ 2\pi m k_B T / h^2\ )^{3/2}$$ [3]

(iv) Hence show that the distribution of particle speeds n(v) in the gas

$$n(v)dv = g(v).f(\varepsilon)\ dv$$

can be written in full as

$$n(v)dv = 4\pi N.(\ m/[2\pi k_B T])^{3/2}.v^2 \exp(-mv^2/2k_B T)dv$$ [3]

(v) Sketch a graph of n(v) versus v and mark the most probable speed v_p. [2]

(vi) Derive an expression for the mean square speed $<v^2>$ and calculate its value for Helium atoms (Relative Atomic Mass = 4) at a temperature of 300K. [3]

(vii) Hence obtain expressions for the energy U and the heat capacity C_V of a mole of He gas. [2]

(viii) What additional excitations can occur in a classical diatomic gas? [2]

(ix) Draw a graph of C_V versus temperature for a diatomic gas showing how the additional excitations affect the shape of the graph and the value of C_V. [4]

Integrals

$I_n = \int_0^\infty x^n \cdot \exp(-bx^2)dx$; $I_n = \left[\frac{(n-1)}{2b}\right] I_{n-2}$

$I_1 = 1/2b$; $I_0 = \frac{1}{2}(\pi/b)^{1/2}$

2.

(b) Phonons propagate in a solid shaped as a cube with side L.

(i) Write conditions for states of phonons to be quantised in terms of wavevector components k_x, k_y, k_z. **[1]**

(ii) Show that, taking account of the different polarizations, the number of phonon states in the range k to k+dk is

$$g(k)dk = 3.V.\frac{4\pi k^2 dk}{(2\pi)^3}$$ **[3]**

where $V = L^3$.

In the Debye model the relation between phonon frequency ν and k is $\nu = kc/[2\pi]$ where c is the velocity of sound.

(iii) Show that the density of phonon states in terms of ν can be written

$$g(\nu)d\nu = \frac{12\pi V.\nu^2 d\nu}{c^3}$$ **[3]**

A solid containing N atoms supports 3N phonons. Show that for such a solid, the cut-off phonon frequency ν_D is given by

$$\nu_D^3 = \frac{3c^3}{4\pi} \cdot \frac{N}{V}$$ **[3]**

The phonon energy can be written

$$U = \int_0^\infty \frac{12\pi\, V\nu^2 \cdot h\nu}{c^3} \cdot \frac{1}{[\exp(h\nu/k_B T) - 1]} \cdot d\nu$$

(iv) Explain the terms $h\nu$ and $\dfrac{1}{[\exp(h\nu/k_BT)-1]}$ **[2]**

(v) Writing $y = (h\nu/k_BT)$ and $y_D = (h\nu_D/k_BT)$ show that

$$U = \frac{12\pi V k_B{}^4 T^4}{c^3h^3}\int_0^{y_D}\frac{y^3dy}{[\exp(y)-1]}$$ **[3]**

(vi) As the temperature $T \to 0$ show that

$$U \to \frac{4\pi^5 V k_B{}^4 T^4}{5c^3h^3}$$ **[2]**

(vii) At high temperature where $k_BT >> h\nu_D$, show that $U \to 3Nk_BT$ **[2]**

(viii) Derive expressions for the heat capacity C_V at these two limits. **[2]**

(ix) Draw a graph of C_V versus T and discuss how well Debye theory represents the actual variation of phonon heat capacity with temperature. **[4]**

Integral

$$\int_0^{\infty}\frac{y^3dy}{[\exp(y)-1]} = \frac{\pi^4}{15}$$

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3. Answer **either** 3(a) **or** 3(b).

(a) Write accounts of **five** of the following topics.

- (i) The Meissner effect in superconductors [5]
- (ii) The magnetic flux quantum [5]
- (iii) The theory that accounts for the critical temperature of a superconductor [5]
- (iv) Experiments that support the two fluid model of liquid He^4 II [5]
- (v) The λ transition in liquid He^4 [5]
- (vi) Superfluid liquid He^3. [5]

(b) Describe **one** method in which liquid helium is used to obtain very low temperatures

The description should include

- (i) the type of liquid helium used [2]
- (ii) the theory of the cooling process [4]
- (iii) a schematic diagram of the equipment [3]
- (iv) the basic procedure [2]
- (v) the starting and final temperatures [2]

Adiabatic demagnetization can be used to cool suitable samples.

Describe this means of cooling – the description should cover

- (vi) the type of material suitable for a sample [2]
- (vii) the process employed [3] 7
- (viii) the theory of the process [4]
- (ix) A sample is magnetized with a field of 10T at a starting temperature of $2x10^{-3}$K. It is then demagnetized leaving only a residual field of $5x10^{-3}$T. Evaluate the final temperature. [3]

CONSTANTS

Speed of light in vacuum	c	=	$3.00 \times 10^{8}\ ms^{-1}$
Permeability of vacuum	μ_0	=	$4\pi \times 10^{-7}\ Hm^{-1}$
Permittivity of vacuum	ε_0	=	$8.85 \times 10^{-12}\ Fm^{-1}$
Elementary charge	e	=	$1.60 \times 10^{-19}\ C$
Planck constant	h	=	$6.63 \times 10^{-34}\ Js$
Avogadro constant	N_A	=	$6.02 \times 10^{23}\ mol^{-1}$
Boltzmann constant	k	=	$1.38 \times 10^{-23}\ JK^{-1}$
Gas constant	R	=	$8.31\ JK^{-1}mol^{-1}$
Unified atomic mass constant	m_u	=	$1.66 \times 10^{-27}\ kg$
		=	$931\ MeVc^{-2}$
Electron mass	m_e	=	$9.11 \times 10^{-31}\ kg$
Proton mass	m_p	=	$1.67 \times 10^{-27}\ kg$
Gravitational constant	G	=	$6.67 \times 10^{-11}\ Nm^2kg^{-2}$
Acceleration due to gravity	g	=	$9.8\ ms^{-2}$
Bohr magneton	μ_B	=	$9.27 \times 10^{-24}\ JT^{-1}$